Complex Engineering Systems
Region stability of switched two-dimensional linear dissipative
Hamiltonian systems with multiple equilibria

--Manuscript Draft--

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<th>Manuscript ID:</th>
<th>CES-2023-13</th>
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<td>Manuscript Title:</td>
<td>Region stability of switched two-dimensional linear dissipative Hamiltonian systems with multiple equilibria</td>
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<tr>
<td>Manuscript Type:</td>
<td>Research Article</td>
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<tr>
<td>Special Issue:</td>
<td></td>
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<tr>
<td>Keywords:</td>
<td>Switched linear systems, dissipative Hamiltonian systems, switched line, regional stability, asymptotic regional stability, multiple equilibrium points, maximum energy function method</td>
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<td>Funding Agency and Grant Number:</td>
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Region stability of switched two-dimensional linear dissipative Hamiltonian systems with multiple equilibria

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Received: date month year First Decision: Revised: Accepted: Published:

Abstract

This paper studies the issues of region stability of switched two-dimensional linear dissipative Hamiltonian systems. Such switched systems are composed of two stable subsystems with two different multiple equilibrium points. Since the two subsystems’ equilibrium points are different, and the subsystems’ state matrices may not commute, it is difficult to address such switched systems. This paper considers the case that the switching path corresponding to the switched systems is a switching line passing through the two different subsystems’ equilibrium points. Firstly, a suitable region containing all the subsystems’ equilibrium points is determined. Then, based on the concept of region stability of switched systems with multiple equilibrium points, this paper proposes some sufficient conditions of region stability and asymptotically region stability for such kind of switched linear dissipative Hamiltonian systems via the maximum energy function method. Finally, two numerical examples and their simulations illustrate the effectiveness of the region stability results obtained in this paper.

Keywords: Switched linear systems, dissipative Hamiltonian systems, switched line, regional stability, asymptotic regional stability, multiple equilibrium points, maximum energy function method

1. INTRODUCTION

Hybrid systems is a kind of complex systems owning both continuous time states and discrete time states. As an important class of hybrid systems, switched systems is composed of finite/infinite subsystems and a switching path/signal, which is used to select one of subsystem to activate at any instants. Since 1990, it has been found that switched systems has been widely used in many physical and practical fields, such as power system, multi-agent technology, hybrid electric vehicle, traffic control systems and so on. Owing to the structural complexity and extensive application of switched systems, it is of great significance to study switched systems. In 2002, D. Liberzon...
and A. S. Morse published a review on the issues of stability analysis of switched systems\textsuperscript{[5]}. The review summarized there are three basic problems on the stability of the switched system as follows. One is the stability of switched systems under any switching paths; Another is the stability of the system under given switching paths; The third is to determine suitable switching paths such that switched systems is stable under the determined switching paths. After that, a large number of researchers have payed more and more attentions to the domain of switched systems and proposed plenty of results for switched systems. Meanwhile, some important methods are developed to analyze the stability of switched systems, such as the common Lyapunov function method (CLF)\textsuperscript{[6]}, the multi-Lyapunov functions method (MLF)\textsuperscript{[7]}, the multi-storage functions method (MSFs)\textsuperscript{[8]} and so on.

Almost all of the above system stability analysis and comprehensive results obtained for switched systems are based upon the same assumption that all subsystems have a common single equilibrium point i. e., the origin, in their common state domain\textsuperscript{[9]-[12]}. However, due to the complexity of the environment and system structures, for each subsystem of switched systems there will be one or more than one equilibrium in its' state domain. Moreover, the equilibrium point of each subsystem in many switched systems may not be the same one. Since there is short of necessary stability analysis methods/techniques, it is indeed a challenge and has a practical significance to investigate switched systems with multiple equilibria. Although it is more difficult to study switched systems with multiple equilibria than that of switched systems with a common single equilibrium point, many researchers have presented some stability results in the literature. For example, in the reference\textsuperscript{[13]} several sufficient conditions of region stability, global asymptotic region stability, and region instability are proposed for switched linear time-invariant systems under arbitrary periodical/quasi-periodical switching paths with respect to a region. The corresponding region contains all the multiple equilibrium points of such kind of switched linear systems with multiple equilibria. The results of the boundedness and practical stability are obtained for switched systems with multiple equilibria in the literature\textsuperscript{[14]} by studying the robustness to external disturbances of such switched systems.

As an effective method the Hamiltonian function method has been widely used to study nonlinear systems. Such method should be based on system energy control and system stability analysis. In practical, a general affine nonlinear system can been converted into a port-controlled Hamiltonian (PCH) systems based on the Hamiltonian realization method\textsuperscript{[15]}. The PCH systems can be accepted a formation of unified mathematical structures for various physical systems such as electric power systems and mechanical systems. The Hamiltonian function of PCH systems has an explicitly physical significance, and often been used as the total energy of physical systems. It is then often selected as an appropriate Lyapunov function. Based on the above advantages of Hamiltonian systems, researchers obtains lots of results of Hamiltonian systems, such as the results of trajectory tracking control\textsuperscript{[16,17]} and $H_{\infty}$ control\textsuperscript{[18]}, etc.

Recently, as an important kind of switched systems, switched Hamiltonian systems (SHS) has also be studied. This may due to such kind of switched systems can be used to model many practical systems composed of finite different modes. In fact, studying the switched Hamiltonian system may induce to provide an effective method for analyzing such class of switched systems. However, comparing to the existing vast results of switched systems with subsystems not being the formation of Hamiltonian systems, there is few results presented for switched Hamiltonian systems in the open literature. Except for the literature\textsuperscript{[5]} studying the stability issues of switched linear Hamiltonian systems, some another literature\textsuperscript{[19,20]} analyze the issues of system stability, system stabilization, and $H_{\infty}$ control of the switched PCH systems, and propose some corresponding results for such kind of switched systems.

It should be pointed out that all the above results obtained for switched PCH systems are also based on the assumption that all the subsystems must has a common single equilibrium point—the origin of the system state space. To the authors’ best knowledge, there are fewer results reported for switched Hamiltonian systems with multiple equilibria in the open literature including the following one. The literature\textsuperscript{[21]} tries to model a power system with a series of faults in the form of switched impulsive Hamiltonian systems (SIHSS) with multiple equilibria (ME), and proposes some necessary and sufficient condition of RS and ARS for the power system with respect to the region via the maximum energy function method that first been introduced in the literature\textsuperscript{[12]}. Especially, it is witnessed that there doesn’t exist any results of switched linear Hamiltonian systems with multiple equilibria in the literature. Therefore, studying switched linear Hamiltonian systems with multiple equilibrium points not only enriches the theoretical results of system analysis and system control of switched systems, but also has important practical significance.

This paper studies the region stability issues of switched linear port-controlled Hamiltonian systems with
multiple equilibrium points. Since it is too difficult to study such a switched system that no only is composed of every subsystem owning more than one different equilibrium points but also the subsystems’ state are not commutative from each other, we just consider in this paper the switched linear Hamiltonian system under three cases as follows.

One is the case that there are only two subsystems. Another is the case that each subsystem has only a unique equilibrium point and the two subsystems’ equilibrium points are different. The third is the case that the involved switching path is the straight line passing through the two subsystems’ equilibrium points. To do that, based on the concepts of region stability ad asymptotic region stability of switched systems with multiple equilibrium points, by means of the maximum energy method we propose the main contributions of this paper, that is, several sufficient conditions of the region stability and asymptotic region stability are given for switched linear PCH systems with respect to a region containing all multiple equilibrium points under the specific switching path. Three numerical examples are carried out to illustrate the effectiveness and practicality of the theoretical results obtained in this paper.

The rest of this paper is organized as follows. Section 2 gives the system expression of switched linear Hamiltonian systems with multiple equilibrium subsystems, definitions, and other preliminaries including notations. Section 3 proposes the main contributions of this paper, i.e., some region stability criteria of switched linear Hamiltonian systems with multiple equilibrium points. Section 4 illustrates numerical examples to show the validity of the obtained new results, which is followed by the conclusion in Section 5.

Notation: \( \mathbb{R} \) and \( \mathbb{R}_+ \) denote respectively the real number field and the positive real number field; \( \mathbb{R}^d \) denotes the \( d \)-dimensional Euclidean space; \( ||.|| \) denotes the norm in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \); \( \mathbb{R}^{l \times s} \) denote the set of \( l \times s \) real matrices, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}_+ \) denotes the set of positive integers; \( d(x,y) = ||x-y|| \), \( d(x,\Omega) = \inf_{y \in \Omega} ||x-y|| \) denote the distances of the two points \( x \) and \( y \) contained in \( \mathbb{R}^d \), the point \( x \in \mathbb{R}^d \) to the compact domain \( \Omega \subset \mathbb{R}^d \), respectively. For a given matrix \( J, J^T \) is the transpose of \( J \), \( J = -J^T \) signifies that \( J \) is skew symmetric. For a given matrix \( R, R > 0 \) and \( R \geq 0 \), represent that \( R \) is positive definite and positive semi-definite. \( \nabla H(x) = \frac{\partial H(x)}{\partial x} \) denotes the gradient of the differentiable function \( H(x) \). Let \( \lambda_i(A) \) be the eigenvalues of the matrix \( A \in \mathbb{R}^{N \times N} \), where \( i = 1, \ldots, N, j = 1, \ldots, n, \) and \( N \in \mathbb{N} \). And \( \lambda_{\min}(A), \lambda_{\max}(A) \) represents the smallest and the largest of all the eigenvalues of the positive definite matrix \( A \), respectively.

### 2. PRELIMINARIES

This section gives the preliminaries needed necessarily for studying switched linear port-controlled Hamiltonian systems with multiple equilibrium points in the next sections. Section 2.1 introduces the switched systems model considered in this paper and some preparatory knowledge and notation. Section 2.2 introduces some relevant definitions and a proposition that will be used in the sequel.

#### 2.1. System description and preliminaries

Consider a switched linear Hamiltonian systems with multiple equilibrium subsystems as follows.

\[
\begin{align*}
\dot{x} &= \left[ J_{\sigma(t)} - R_{\sigma(t)} \right] \nabla H_{\sigma(t)}(x), \\
x(t_0) &= x_0,
\end{align*}
\]

for all \( t \geq t_0 \),

where \( x = [x_1, x_2]^T \in \mathbb{D} \) represents the state of the system, and \( \mathbb{D} \) is the common domain of all the subsystems of system (1). The map \( \sigma(t) : [t_0, +\infty) \to \Lambda := \{1, 2\} \) is a piecewise right continuous constant step function, which is called to be a switching path or a switching rule. The value of the function \( \sigma(t) = i \), where \( i = 1, 2 \), means that the subsystem \( i \) is activated at the time instant \( t \). The matrices \( J_i \in \mathbb{R}^{2 \times 2} \) and \( R_i \in \mathbb{R}^{2 \times 2} \) are all constant matrices. Moreover, \( J_1^T = -J_2 \) and \( R_1^T = R_2 \). The notation \( \nabla H_i(x) = \frac{\partial H_i(x)}{\partial x} \) is the gradient of the energy function \( H_i(x) \) of subsystem \( i \). The function \( H_i(x) \), \( i = 1, 2 \), satisfy the following two: (1) they are continuous and differentiable; (2) \( H_i(x) > 0 \) for any \( x \in \mathbb{D} - \{x^e1, x^e2\} \), in which \( x^e1 \) satisfying \( H_1(x^e1) = 0 \) is a unique equilibrium point of subsystem \( i \). Moreover, the two subsystems’ equilibrium points \( x^e1 \) and \( x^e2 \) are different from each other. In fact, the system (1) considered in this paper is a switched two-dimensional linear systems only consisted with two subsystems.

#### 2.1.1 Switching line

To facilitate the subsequent analysis, the switching path \( \sigma(t) \) of system (1) can be expressed as \( \sigma(t) = i_m \in \{1, 2\} \), \( t \in \{t_m, t_{m+1}\}, m = 0, 1, \ldots \), and then we express by \( x(t) := x(t; t_0, x_0, \sigma) \) the trajectory of system ((1)) under the switching path \( \sigma(t) \) starting from the initial state \( x(t_0) \) at the initial time \( t_0 \). Express by \( \{x_m\}_{m=0}^{\infty} \) the switching state
sequence and by \( \{ t_m \}_{m=0}^{\infty} \) the switching time sequence. Express by \( \{ H_m (x_m) \}_{m=0}^{\infty} \) the switching energy sequence and by \( \{ i_m \}_{m=0}^{\infty} \) the switching index sequence, where \( x_m = x(t_m), \ i_m \neq i_{m+1}, \ \forall m \in \mathbb{N} \). Express by \( t_m, \ \forall l \in \mathbb{N}, \) the switching time when the \( l \)-th subsystem is switched on.

Throughout this paper, the switching path \( \sigma(t) \) governed system (1) is assumed to satisfy the following four cases:

- Switching phenomena of system (1) only appear on a straight line that passing through the two equilibrium points of the first subsystem and the second subsystem of system (1). Such a straight line is called a switching line in this paper and denoted by \( l_1 \).
- Any one of all the subsystems cannot be excluded from those activated subsystems as the time goes to infinity.
- The switching times of the switching path \( \sigma(t) \) is finite number over any finite time intervals.
- The whole state trajectory \( x(t) \) of system (1) is continuous at any switching time instants \( t_m \), where any \( m \in \mathbb{N} \).

Moreover, let \( x^{e_1} = [x_1^{e_1}, x_2^{e_1}]^T \) and \( x^{e_2} = [x_1^{e_2}, x_2^{e_2}]^T \) be the equilibrium points of the first subsystem and the second subsystems, respectively. The state or trajectory of system ((1)) is denoted by \( x(t) = [x_1, x_2]^T \). The initial state of the system ((1)) at the initial time \( t_0 \) is denoted by \( x(t_0) = [x_1^0, x_2^0]^T \). Then, for the two equilibrium points \( x^{e_1} \) and \( x^{e_2} \) of the first and second subsystems there exists a straight line passing through them. The straight line is denoted by \( l_1 \) in this paper. Without loss of generality, for the straight line \( l_1 \) there are two cases as follows.

1. If the two subsystems’ equilibrium points \( x^{e_1} \) and \( x^{e_2} \) satisfying \( x^{e_2} > x^{e_1} \), then the switching line \( l_1 \) can be expressed as

   \[
   l_1 : x_2 = \frac{x_2^{e_2} - x_2^{e_1}}{x_1^{e_2} - x_1^{e_1}} x_1 + \frac{x^{e_2} x_1^{e_1} - x^{e_1} x_1^{e_2}}{x_1^{e_2} - x_1^{e_1}} = k x_1 + b,
   \]

   where the two parameters of \( k \) and \( b \) are respectively as follow.

   \[
   k = \frac{x_2^{e_2} - x_2^{e_1}}{x_1^{e_2} - x_1^{e_1}} \quad \text{and} \quad b = \frac{x_1^{e_2} x_2^{e_1} - x_2^{e_2} x_1^{e_1}}{x_1^{e_2} - x_1^{e_1}}.
   \]

2. If the two subsystems’ equilibrium points \( x^{e_1} \) and \( x^{e_2} \) satisfying the following: \( x^{e_1} = x^{e_2} \), i.e., the slope of the line \( k = \infty \), then the switching line \( l_1 \) can be expressed

   \[
   l_1 : x_1 = x_1^{e_1} = x_1^{e_2}.
   \]

Based on the above, we know that the trajectory \( x(t) \) of system ((1)) under the switching line \( l_1 \) passes into and out of the switching line \( l_1 \) as time goes to infinity. This implies that all the switching state sequence \( \{ x_m \}_{m=0}^{\infty} \) are situated in the switching line \( l_1 \). Then, all the switching energy sequence \( \{ H_m (x_m) \}_{m=0}^{\infty} \) and the switching time sequences \( \{ t_m \}_{m=0}^{\infty} \) corresponding to the switching state sequence \( \{ x_m \}_{m=0}^{\infty} \) are also related to the switching line \( l_1 \).

2.1.2 The Hamiltonian functions

In this paper, the Hamiltonian functions of the two subsystems of system (1) are assumed to be the following quadratic forms:

\[
H_i (x) = \frac{1}{2} (x - x^{e_i})^T Q_i (x - x^{e_i}), \quad i = 1, 2, \tag{5}
\]

where \( Q_1 \) and \( Q_2 \) are two positive definite matrices.

Since the switching states \( x_m \) for any \( m \in \mathbb{N} \) are all on the switching line \( l_1 \), we know from (5) that all the switching states satisfy the following formula:

\[
L_i (x_1) = g_i (x_1 - x_1^{e_i})^2, \quad i = 1, 2, \tag{6}
\]

where \( Q_i, \ i = 1, 2, \) are the same positive definite matrices in (5), and

\[
g_i := \frac{1}{2} [1 \ k] Q_i \left[ \begin{array}{c} 1 \\ k \end{array} \right] > 0, \quad i = 1, 2, \tag{7}
\]
where $k$ is defined as in (3).

For the case that $k = \infty$, one knows from the Hamiltonian functions expressed in ((5)) that all the switching states $x_m$ for any $m \in \mathbb{N}$ satisfy the following formula

$$M_i(x_2) = \delta_i(x_2 - x_{r}^i)^2,$$

(8)

where $Q_i$, $i = 1, 2$, are the same two positive definite matrices in (5) and

$$\delta_i := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Q_i \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0.$$

(9)

### 2.2. Some definitions and propositions

This subsection refers some definitions and gives a proposition that are needed to analyze region stability of system (1) in the next section below.

**Definition 1** (The maximum energy function)\textsuperscript{[12]}. The following function $H(x)$ is called the maximum energy function of system (1)

$$H(x) := \max \{H_1(x), H_2(x)\}, \text{ for all } x \in \mathbb{D}$$

(10)

**Definition 2** (The maximum switching energy sequence)\textsuperscript{[12]}. The following sequence is known as the maximum switching energy sequence of the switching path $\sigma(t)$

$$\{H(x_m)\}_{m=0}^{\infty} := \left\{ \max \{H_1(x_m), H_2(x_m)\} \right\}_{m=0}^{\infty}$$

(11)

**Proposition 1** Consider system (1). The subsystems’ Hamiltonian functions $H_1(x)$ and $H_2(x)$ satisfy the following formula

$$\frac{1}{2} \lambda_{\min}(Q_i) d^2(x, x^{e_i}) \leq H_i(x) \leq \frac{1}{2} \lambda_{\max}(Q_i) d^2(x, x^{e_i}), \forall x \in \mathbb{D} - \{x^{e_i}\}, i = 1, 2,$$

(12)

where $\lambda_{\min}(Q_i)$ and $\lambda_{\max}(Q_i)$ are respectively the minimum and maximum eigenvalues of the matrix $Q_i$ for $i = 1, 2$.

**Proof:** The Hamiltonian functions $H_i(x)$, $i = 1, 2$, of subsystems are assumed to be expressed as in (5). Letting $X_i = (x - x^{e_i})$, one obtains from (5) that the following equation holds.

$$F_i(X_i) = \frac{1}{2} X_i^T Q_i X_i, \quad i = 1, 2.$$  

(13)

Since $Q_i$ is a real symmetric matrix of positive definite, there is an orthogonal matrix $P_i$ such that there is an orthogonal transformation $X_i = P_i y_i$ changing $F_i(X_i)$ in (13) into a standard form as follows.

$$O_i(y_i) = \frac{1}{2} y_i^T P_i^T Q_i P_i y_i = \frac{1}{2} y_i^T Z_i y_i, \quad i = 1, 2,$$

(14)

where

$$y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \end{bmatrix}, \quad Z_i = \begin{bmatrix} \lambda_{i1} & 0 \\ 0 & \lambda_{i2} \end{bmatrix},$$

in which $\lambda_{i1}$ and $\lambda_{i2}$ are the two eigenvalues of the definite positive matrix $Q_i$, $i = 1, 2$.

Next, calculating straightly yields the square norm of the vector $X_i$ as follows.

$$\|X_i\|^2 = X_i^T X_i = (P_i y_i)^T (P_i y_i) = y_i^T y_i = \|y_i\|^2, \quad i = 1, 2.$$  

(15)

We know from (14) and (15) that the following two hold.

$$O_i(y_i) = \frac{1}{2} y_i^T Z_i y_i = \frac{1}{2} (\lambda_{i1}(y_{i1})^2 + \lambda_{i2}(y_{i2})^2) \geq \frac{1}{2} \lambda_{\min}(Z_i) \|y_i\|^2$$

(16)
and

\[ O_i(y_i) = \frac{1}{2} y_i^T Z_i y_i = \frac{1}{2} (A_{1i}(y_{1i})^2 + A_{2i}(y_{2i})^2) \leq \frac{1}{2} \lambda_{max}(Z_i) \| y_i \|^2. \]  

(17)

It can be obtained from (5), (13), (15), (16), and (17) that

\[ H_i(x) = O_i(y_i). \]  

(18)

It follows from (18) that for any \( x \in \mathbb{D} - \{x^e\} \),

\[ \frac{1}{2} \lambda_{min}(Q_i) d^2(x, x^e) \leq H_i(x) \leq \frac{1}{2} \lambda_{max}(Q_i) d^2(x, x^e), \quad i = 1, 2, \]

which is exact (12). The proof of Proposition 1 is thus completed.

Since every subsystems of system (1) has a unique equilibrium point \( x^e_i \), \( i = 1, 2 \), and the maximum energy functions in Definition 1 are continuous everywhere in \( \mathbb{D} \), there exists a unique compact region that is defined as

\[ \Omega := \left\{ z \in \mathbb{D} \mid H_1(z) \leq g_1(x^e_1 - x^e_2)^2 \right\} \bigcup \left\{ z \in \mathbb{D} \mid H_2(z) \leq g_2(x^e_1 - x^e_2)^2 \right\} \subseteq \mathbb{D}, \quad \text{as } x^e_1 \neq x^e_2. \]  

(19)

or

\[ \Psi := \left\{ z \in \mathbb{D} \mid H_1(z) \leq \delta_1(x^e_2 - x^e_1)^2 \right\} \bigcup \left\{ z \in \mathbb{D} \mid H_2(z) \leq \delta_2(x^e_1 - x^e_2)^2 \right\} \subseteq \mathbb{D}, \quad \text{as } x^e_1 \neq x^e_2. \]  

(20)

Similar to that of region stability defined in the reference\textsuperscript{17}, based on the region \( \Omega \) in (19) or the region \( \Psi \) in (20) we introduce the concept of region stability for system (1) as follows.

Definition 3 Consider system (1) with the region \( \Omega \) defined in (19) or \( \Psi \) defined in (20) under a special kind of switching path \( \sigma(t) \), i.e., the switching line \( l_1 \) in (2). System (1) under the switching line \( l_1 \) in (2) is said to be

- Region stable with respect to the region \( \Omega \) in (19) or the region \( \Psi \) in (20), if for \( \forall \varepsilon > 0, \exists \delta := \delta(\varepsilon) > 0 \) such that the following formula holds for any \( x_0 \),

\[ d(x_0, \Omega) < \delta \implies d(x(t), \Omega) < \varepsilon, \quad t \in [t_0, +\infty) \]  

(21)

or

\[ d(x_0, \Psi) < \delta \implies d(x(t), \Psi) < \varepsilon, \quad t \in [t_0, +\infty). \]  

(22)

- Asymptotically region stable with respect to the region \( \Omega \) in (19) or the region \( \Psi \) in (20), if both (21) or (22) and the following limit hold

\[ \lim_{t \to +\infty} d(x(t), \Omega) = 0 \quad \text{or} \quad \lim_{t \to +\infty} d(x(t), \Psi) = 0. \]  

(23)

3. REGION STABILITY ANALYSIS

This section will study the stability issue of switched two-dimensional linear Hamiltonian systems with multiple equilibria. Based on the concept of region stability defined in Section 2, we propose several sufficient conditions of region stability and asymptotic region stability for system (1), respectively.

3.1. Some lemmas

This subsection introduces some lemmas that will be used in the next subsection. Firstly, it can be obtained from Proposition 1 that the following result holds.

Lemma 1 Consider system (1) under the switching line (2). The Hamiltonian functions \( H_1(x) \), \( H_2(x) \), and the maximum energy function \( H(x) \) satisfy the following inequation.

\[ \frac{1}{2} \sigma d^2(x, \Omega) \leq H_1(x) \leq H(x) \leq \frac{1}{2} \beta d^2(x, \Omega), \quad \forall x \in \mathbb{D} - \Omega, \quad i = 1, 2, \]  

(24)

where \( \sigma = \min\{\lambda_{min}(Q_1), \lambda_{min}(Q_2)\} \), \( \beta = \max\{\lambda_{max}(Q_1), \lambda_{max}(Q_2)\} \), and the region \( \Omega \) and the maximum energy function \( H(x) \) are respectively defined in (19) and (10).
**Proof:** It is easy to see from the maximum energy function defined in (10) of Definition 1 and the equation (12) in Proposition 1 and Condition (19) that Lemma 1 holds true. 

**Lemma 2** Consider system (1) under the switching line (2). If $R_i \geq 0$, then for the trajectory $x(t) \in D - \Omega$, $\forall t \in [t_m, t_{m+1})$, the two Hamiltonian functions $H_1(x)$ and $H_2(x)$ have the same variant trend of the properties of either monotonous increase or monotonous decrease at all the switching states $x_m$, where any $m \in \mathbb{N}$. That is, for all the switching states $x_m, x_{m+1} \in D - \Omega$, $\Delta H_i := H_i(x_m) - H_i(x_{m+1})$ and $\Delta H_j := H_j(x_m) - H_j(x_{m+1}), i = 1, 2$, satisfy

\[\text{sign}(\Delta H_i) = \text{sign}(\Delta H_j),\]  

(25)

where $\text{sign}(\cdot)$ denotes the sign function.

**Proof:** Since for $x(t) \in D - \Omega$, $\forall t \in [t_m, t_{m+1})$, all the switching states $x_m$ are situated in the switching line $l_1$ in (2), the following formula holds

\[H_{im}(x_m) = L_{im}(x_{1,m}) = g_{im}(x_{1,m} - x_{1}^e)^2,\]  

(26)

where $x_{1,m}$ and $x_{1}^e$ denote the first elements of the switching state $x_m$ and the equilibrium point $x^e$, respectively; And the parameters $g_{im}$ are defined as follows.

\[g_{im} := \frac{1}{2} \left[ \begin{array}{c} 1 \\ k \end{array} \right]^T Q_{im} \left[ \begin{array}{c} 1 \\ k \end{array} \right] > 0, \quad i_m \in \{1, 2\}, \quad m \in \mathbb{N}.\]

It follows from the fact that the trajectory $x(t) \in D - \Omega$ and $R_{im}(x) \geq 0$ that for $t \in [t_m, t_{m+1}), i_m = 1, 2, m \in \mathbb{N}$, the Hamiltonian function $H_{im}(x)$ satisfies

\[\dot{H}_{im}(x) = -(x - x^e)^T Q_{im} R_{im} Q_{im}(x - x^e) \leq 0.\]  

(27)

One obtains from (27) that

\[H_{im}(x_{m+1}) \leq H_{im}(x_m), \quad i_m = 1, 2, \text{ for any } m \in \mathbb{N}.\]  

(28)

Since the first element $x_{1}^e$ of the equilibrium point $x^e$ are all contained in the region $\Omega$, we know from (19), (24), (26), and (28) that for any $m \in \mathbb{N}$ and $i_m = 1, 2$,

\[L_{im}(x_{1,m+1}) \leq L_{im}(x_{1,m}) \iff d(x_{1,m}, x_{1}^e) > d(x_{1,m+1}, x_{1}^e) \iff d(x_m, \Omega) > d(x_{m+1}, \Omega).\]  

(29)

It is then obtained from (26) and (29) that

\[H_{im}(x_{m+1}) \leq H_{im}(x_m) \iff d(x_m, \Omega) > d(x_{m+1}, \Omega).\]  

(30)

One obtains from (26) and (30) that for all $i_m, i_s \in \Lambda$ and $i_m \neq i_s$, the following holds.

\[d(x_m, \Omega) > d(x_{m+1}, \Omega) \implies L_{i_s}(x_{1,m+1}) \leq L_{i_s}(x_{1,m}) \implies H_{i_s}(x_{m+1}) \leq H_{i_s}(x_m).\]  

(31)

From which and (30) it follows that

\[\text{sign}(\Delta H_i) = \text{sign}(\Delta H_j),\]  

(32)

which is indeed the equation (25). Thus, Lemma 2 holds true. 

**Lemma 3** [22] Let

\[a_{n}^{\text{min}} := \min_{i \in \Lambda}(a_{n}^{i}), \quad n = 1, 2, \ldots\]  

(33)

and

\[a_{n}^{\text{max}} := \max_{i \in \Lambda}(a_{n}^{i}), \quad n = 1, 2, \ldots\]  

(34)

If the two infinite sequences $\{a_n^{i}\}_{n=1}^{+\infty}, i = 1, 2$, are both monotonically decreasing/increasing, then $\{a_{n}^{\text{min}}\}_{n=1}^{+\infty}$ and $\{a_{n}^{\text{max}}\}_{n=1}^{+\infty}$, are also monotonically decreasing/increasing sequence.
Lemma 4  [21] System (1) under the switching line $l_1$ in (2) is region stable with respect to the region $\Omega$ in (19) if and only if for any $x_0 \in \mathcal{D}$, the following holds:

$$H_{lm}(x_m) \leq CH(x_0), \quad i_m \in \Lambda, \quad m \in \mathbb{N} \quad (35)$$

where the $i_m = \sigma(t_m) \in \{1, 2\}$; The parameter $C$ is a constant; And $H(x)$ is defined as in Definition 1.

Remark 1 If the region $\Omega$ in (19) is replaced by the region $\Psi$ in (20), then all Lemmas 1-4 hold too.

3.2. Regional stability results

Based on Definition 3, we obtain from Proposition 1, Lemmas 1-4 that two main results of this paper are proposed in series as follows.

For the horizontal and vertical ordinates of the two subsystems’ equilibrium points $x^e_1$ and $x^e_2$, there are the following two cases: (1) $x^e_1 \neq x^e_2$; (2) $x^e_1 = x^e_2$ and $x^e_1 \neq x^e_2$. For the former we propose the following regional stability result of system (1) under the switching line $l_1$ in (2) as follows.

Theorem 1 Consider system (1) with the compact region $\Omega$ in (19) and the switching line $l_1$ in (2). For the case that $x^e_1 \neq x^e_2$, system (1) under the switching line $l_1$ in (2) is

(i) region stable with respect to the region $\Omega$ in (19), if $R_1 \geq 0$, $R_2 \geq 0$, and

$$\frac{\beta}{\alpha} \geq \max \left\{ 1, 4 \left( \frac{x^e_2 - x^e_1}{v_1 - x^e_1} \right)^2, 4 \left( \frac{x^e_2 - x^e_1}{v_1 - x^e_2} \right)^2 \right\}, \quad (36)$$

where $\alpha = \min \{\lambda_{min}(Q_1), \lambda_{min}(Q_2)\}$, $\beta = \max \{\lambda_{max}(Q_1), \lambda_{max}(Q_2)\}$, and $v_1$ is the intersection point of the two parabolic curves $L_1(x_1)$ and $L_2(x_1)$ in (6) over the interval $(x^e_1, x^e_2)$ in the horizontal axis, i.e.,

$$v_1 = \begin{cases} \frac{g_1(x^e_1 - x^e_2)^2}{g_1x^e_1 + g_2x^e_2}, & \text{as } g_1 \neq g_2, \\ \frac{x^e_1 + x^e_2}{2}, & \text{as } g_1 = g_2, \end{cases} \quad (37)$$

where $g_1$ and $g_2$ are defined in (7).

(ii) asymptotic region stable with respect to the region $\Omega$, if both the matrices $R_1 > 0$ and $R_2 > 0$ and the condition (36) is satisfied.

Proof: Without loss of generality, we just show that the two statements (i) and (ii) hold for the case that $x^e_1 < x^e_2$. As for the case that $x^e_1 > x^e_2$ it is similar to show that the two statements also hold true.

For any trajectory $x(t)$ of system (1) under the switching line $l_1$ in (2) starting from any initial state $x_0$ at the initial time $t_0$, there are the following three cases that should be considered: (a) The system’s trajectory $x(t) \in \mathcal{D} - \Omega$, $\forall t > t_0$; (b) The system’s trajectory $x(t)$, $\forall t > t_0$, is always contained in $\Omega$; (c) The system’s trajectory $x(t)$, $\forall t > t_0$, is neither always contained in $\mathcal{D} - \Omega$ nor always contained in $\Omega$.

(1) We show the conclusion of Theorem 1 holds for the case that $g_1 \neq g_2$.

Firstly, we will show that the statement (i) holds for Case (a). Since the two subsystems’ Hamiltonian functions $H_1(x)$ and $H_2(x)$ are both continuous everywhere in $\mathcal{D}$, the maximum energy function $H(x)$ defined in Definition 1 is also continuous everywhere in $\mathcal{D}$. We obtain from the ordinary differential equations (ODEs) (1) and (36) that for any $t \in [t_m, t_{m+1})$ and $m \in \mathbb{N}$, the following holds,

$$H_{lm}(x) = -(x - x^{e_{lm}})^T Q_{lm} R_{lm} Q_{lm} (x - x^{e_{lm}}) \leq 0. \quad (38)$$

It follows from (38) that

$$H_{lm}(x(t_{m+1})) \leq H_{lm}(x(t_m)), \quad i_m = 1, 2, \quad \text{for any } m \in \mathbb{N}. \quad (39)$$

By Lemma 2 one knows from (39) that for $i_s \neq i_m \in \{1, 2\}$,

$$H_{is}(x(t_{m+1})) \leq H_{is}(x(t_m)), \quad \text{for any } m \in \mathbb{N}. \quad (40)$$
It can be obtained from (32),(36) and (40) that the infinite number sequence \( \{H_i(x(t_m))\}_{m=0}^{\infty} \) is monotone decreasing. Then, it can be known from Lemma 3 that the switching maximum energy sequence \( \{H(x_m)\}_{m=0}^{\infty} \) is also monotone decreasing.

Based on the above analysis, one obtains from (10) in Definition 1 and the fact that the trajectory \( x(t) \in \mathbb{D} - \Omega \), for all \( t \in [t_m,t_{m+1}) \), \( i_m = 1,2 \), and any \( m \in \mathbb{N} \) that

\[
H_{i_m}(x(t_m)) \leq H(x(t_m)) \leq H(x(t_0)). 
\]

(41)

It is easy to see from (41) that

\[
H_{i_m}(x(t_m)) \leq H(x(t_0)). 
\]

(42)

From which and (36) we know that

\[
H_{i_m}(x_m) \leq \frac{\beta}{\alpha} H(x_0), \quad i_m = 1,2, \quad \text{for any} \quad m \in \mathbb{N}. 
\]

(43)

It follows from (43) that (35) in Lemma 4 is satisfied for system (1). Then, by Lemma 4 we know that system (1) under the switching line \( l_1 \) in (2) is region stable with respect to the region \( \Omega \) for Case (a).

Secondly, we will show that statement (i) also holds for case (b). For \( g_1 \) and \( g_2 \) in (2) there are the following two relationships: \( g_1 = g_2 \) and \( g_1 \neq g_2 \). We first consider the case that \( g_1 \neq g_2 \). In this case, letting \( L_1(x_1) = L_2(x_1) \) one obtains from (6) that

\[
(g_1 - g_2)(x_1)^2 + (2g_2x_1^2 - 2g_1x_1^2)x_1 + g_1(x_1^2)^2 - g_2(x_1^2)^2 = 0. 
\]

(44)

From which it can be known that the discriminant of the roots of the equation (44) is as follows.

\[
\Delta = 4g_1g_2(x_1^2 - x_2^2)^2 > 0, 
\]

(45)

which means that two curves \( L_1(x_1) \) and \( L_2(x_1) \) have two intersection points.

It follows from (45) that the equation (44) have the following two solutions:

\[
v_1 = \frac{(g_1x_1^2 - g_2x_2^2) + \sqrt{g_1g_2(x_1^2 - x_2^2)^2}}{g_1 - g_2}, 
\]

(46)

and

\[
v_2 = \frac{(g_1x_1^2 - g_2x_2^2) - \sqrt{g_1g_2(x_1^2 - x_2^2)^2}}{g_1 - g_2}, 
\]

(47)

where \( v_1 \) and \( v_2 \) denote respectively the two intersection points of the two curves \( L_1(x_1) \) and \( L_2(x_1) \) satisfying \( v_1 \in (x_1^2, x_2^2) \) and \( v_2 \in (-\infty, x_1^2) \cup (x_2^2, +\infty) \).

Next, we will find the intersection points of the switching line \( l_1 \) passing through the the boundary of the region \( \Omega \). To do that, we obtain from \( L_1(x_1) = L_1(x_1^2) \) that

\[
(x_1)^2 - 2x_1^1x_1 - (x_1^2)^2 + 2x_1^1x_1^2 = 0. 
\]

(48)

It can be obtained from (48) that the discriminant of the roots of the equation (48) is as follows.

\[
\Delta = 4(x_1^2 - x_1^2)^2 > 0. 
\]

(49)

It then follows from (49) that the solutions of the equation (48) are as follows.

\[
x_1 = x_1^1 \pm (x_1^2 - x_1^1). 
\]

(50)

The two intersection points can be denoted by \( p_1 = x_1^2 \) and \( p_2 = 2x_1^1 - x_1^2 \).
Similarly, solving $L_2(y_1) = L_2(x_1^{e1})$ yields that

$$y_1 = x_1^{e2} \pm (x_1^{e2} - x_1^{e1})$$

(51)

The two intersections are then denoted by $p_3 = x_1^{e1}$ and $p_4 = 2x_1^{e2} - x_1^{e1}$, respectively.

One obtains from (50) and (51) that the switching line $l_1$ in (2) passing through the largest point and the smallest point of the boundary of the region $\Omega$. The two maximum and minimum points are respectively denoted by

$$p_{max} = \max_{i \in [4]} \{p_i\} = 2x_1^{e2} - x_1^{e1} \text{ and } p_{min} = \min_{i \in [4]} \{p_i\} = 2x_1^{e1} - x_1^{e2}.$$  

(52)

In the following, we consider the minimum value of the maximum energy function $H(x)$ over the region $\Omega$.

(C1.) As $v_2 \in (-\infty, x_1^{e1})$, the maximum energy function $H(x)$ is as follows.

$$H(x) = \begin{cases} 
L_2(x_1) = g_2(x_1 - x_1^{e2})^2, & \text{as } x_1 \in (v_2, v_1] \\
L_1(x_1) = g_1(x_1 - x_1^{e1})^2, & \text{as } x_1 \in [v_1, +\infty).
\end{cases}$$

(53)

From (53) one obtains that the $L_2(x_1)$ is continuous over the interval $(v_2, v_1]$ and $\frac{dL_2(x_1)}{dx_1} < 0$, so the minimum value of $L_2(x_1)$ over the interval $(v_2, v_1]$ is $L_2(v_1)$.

Similarly, we know from (53) that the $L_1(x_1)$ is continuous over the interval $(v_1, +\infty)$ and $\frac{dL_1(x_1)}{dx_1} > 0$, so the minimum value of $L_1(x_1)$ over the interval $(v_1, +\infty)$ is $L_1(v_1)$.

It is obvious from (44) and (46) that

$$L_1(v_1) = L_2(v_1)$$

(54)

From (53) and (54) it shows that the minimum value of $H(x)$ under the case (C1.) is as follows.

$$\inf \left\{ H(x) : x = [x_1 \ x_2]^T \in \mathbb{R}^2 \text{ and } x_1 \geq v_2 \right\} = \min \left\{ L_1(v_1), L_2(v_1) \right\} = L_1(v_1) = L_2(v_1),$$

(55)

where $v_1$ is expressed in (46).

(C2) As $v_2 \in (x_1^{e2}, +\infty)$, the maximum energy function $H(x)$ is as follows.

$$H(x) = \begin{cases} 
L_1(x_1) = g_1(x_1 - x_1^{e1})^2, & \text{as } x_1 \in [v_1, v_2] \\
L_2(x_1) = g_2(x_1 - x_1^{e2})^2, & \text{as } x_1 \in (-\infty, v_1].
\end{cases}$$

(56)

It can be obtained from (56) that the $L_2(x_1)$ is continuous over the interval $(-\infty, v_1)$ and $\frac{dL_2(x_1)}{dx_1} < 0$. Then, the minimum value of the maximum energy function $H(x) = L_2(x_1)$ on the interval $(-\infty, v_1]$ is $L_2(v_1)$.

Similarly, we know that the $L_1(x_1)$ is continuous over the interval $[v_1, v_2]$ and $\frac{dL_1(x_1)}{dx_1} > 0$. Then, the minimum value of the maximum energy function $H(x) = L_1(x_1)$ over the interval $[v_1, v_2]$ is $L_1(v_1)$.

Therefore, it can be obtained from (53) and (54) that the minimum value of the maximum energy function $H(x)$ under the case (C2) is as follows.

$$\inf \left\{ H(x) : x = [x_1 \ x_2]^T \in \mathbb{R}^2 \text{ and } x_1 \leq v_2 \right\} = \min \left\{ L_1(v_1), L_2(v_1) \right\} = L_1(v_1) = L_2(v_1).$$

(57)

Then, one obtains from (55) and (57) that the minimum value of the maximum energy function $H(x)$ over the region $\Omega$ is as follows.

$$\min \left\{ H(x) : x \in \Omega \right\} = L_1(v_1) = L_2(v_1).$$

(58)
It then follows from (36) and the condition that $R_1 > 0$ and $R_2 > 0$ that
\[
\frac{\beta}{\alpha} g_1(v_1 - x_1^e)^2 \geq g_1(p_{max} - x_1^e)^2
\] (59)
and
\[
\frac{\beta}{\alpha} g_2(v_1 - x_1^e)^2 \geq g_2(p_{min} - x_1^e)^2,
\] (60)
where $p_{max}$ and $p_{min}$ are same as in (52).

From (53), (54), (56), (59), and (60) one obtains that
\[
\frac{\beta}{\alpha} \min \{H(x) : x \in \Omega\} \geq \max \{L_1(p_{max}), L_2(p_{min})\},
\] (61)
where $\max \{L_1(p_{max}), L_2(p_{min})\}$ denotes the maximum value of $H(x)$ over the region $\Omega$.

It is obvious from (58) and (61) that for any switching states $x_m$ the following holds.
\[
\frac{\beta}{\alpha} H(x_0) \geq H_{im}(x_m), \ i_m = 1, 2, \text{ for any } m \in \mathbb{N},
\] (62)
which implies that (35) in Lemma 4 is satisfied. By Lemma 4 we know that for Case (b) system (1) under the switching line $l_1$ in (2) is region stable with respect to the region $\Omega$.

Thirdly, we show that Statement (i) also holds for Case (c) as follows. In such case, the trajectory $x(t)$ of system (1) is not always in the region $D - \Omega$. For any small enough $\varepsilon > 0$, there is a time interval sequence $\{[t_m, t_{m+1}]\}_{m=1}^{+\infty}$, such that the states $x_m = x(t_m; t_0, x \in \Omega)$ are all on the boundary of the set $\Omega_e := \{z \in D : d(z, \Omega) \leq \varepsilon\}$. We insert $x_{m'}$ and $x_{m+1}$ into the switching state sequence $\{x_m\}_{m=0}^{+\infty}$. Consequently, the times $t_m$ and $t_{m+1}$ are inserted into the switching time sequence $\{t_m\}_{m=0}^{+\infty}$, and the indexes $l_{m'}$ and $l_{m+1}$ are all inserted into the switching index sequence $\{l_m\}_{m=0}^{+\infty}$.

Based on the above statement, we know that there is a time interval $[t_{m'}, t_{m'+1}]$ for $m' \in \mathbb{N}$, such that for
\[
x(t: t_{m'}, x_{m'}, l_{m'}) \in D - \text{int}(\Omega_e) \subset D - \Omega
\] (63)
It is known from (63) that Statement (i) is true for the trajectory of $x(t: t_{m'}, x_{m'}, l_{m'})$ of system (1) contained in $D - \text{int}(\Omega_e)$. Thus the proof is similar to that proof of Case (a). On the other hand, it is obvious that as $\varepsilon \to 0$, The state trajectory $x(t)$ contained in the region $\Omega_e - \Omega$ will gradually go into the region $\Omega$. Based on the above, we know that Statement (i) holds too.

Finally, Statement (ii) will be shown under the following two situations.

(S1) As $v_2 \in (-\infty, x_1^e)$ and $x(t) \in D - \Omega$, for all $t \in [t_m, t_{m+1}]$ and any $m \in \mathbb{N}$, the values of the maximum energy function $H(x)$ at the switching states $x_m = [x_{1,m}, x_{2,m}]^T \in \mathbb{R}^2, m \in \mathbb{N}$, are as follows.
\[
H(x_m) = \begin{cases} 
L_2(x_{1,m}) = g_2(x_{1,m} - x_1^e)^2, & x_{1,m} \in (v_2, v_1] \\
L_1(x_{1,m}) = g_1(x_{1,m} - x_1^e)^2, & x_{1,m} \in (-\infty, v_2) \cup (v_1, +\infty)
\end{cases}
\] (64)
The subsystem $i_m$ is activating when any $t \in [t_m, t_{m+1}]$ and $x(t) \in D - \Omega$. And it is obvious from Condition $R_1 > 0$ and $R_2 > 0$ of Statement (ii) that for any $i_m = 1, 2,$
\[
H_{im} = -(x_m - x_i^e)^T Q_{im} R_{im} Q_{im} (x_m - x_i^e) < 0, i_m = 1, 2, m \in \mathbb{N}.
\] (65)
We know from (65) that
\[
H_{im}(x(t_{m+1})) < H_{im}(x(t)) < H_{im}(x(t_m)), \ i_m = 1, 2, m \in \mathbb{N}.
\] (66)
From Lemma 2, the condition of $R_1 > 0$ and $R_2 > 0$, (33), (34), (36), and (38) one can show that the two sequences of $\{L_1(x_{1,m})\}_{m}^{\infty}$ and $\{L_2(x_{1,m})\}_{m}^{\infty}$ both monotonically decrease over the interval $(-\infty, x_1^{e_1}) \cup (x_1^{e_2}, +\infty)$. Then, the switching maximum energy sequence of $\{H(x_m)\}_{m}^{\infty}$ also decreases monotonically over the following time interval: $(-\infty, x_1^{e_1})$.

On the other hand, one obtains from Lemma 2 and (66) that
\[
H(x(t_{m+1})) < H(x(t)) < H(x(t_m)), \quad i_m = 1, 2, \quad m \in \mathbb{N}.
\]  
(67)

Then, as $m \rightarrow \infty$ and all the horizontal ordinates $x_{1,m}$ of switching states $x_m$ are contained in the interval $(-\infty, p_{min}) \subset (-\infty, x_1^{e_1})$, where $p_{min}$ and $x_1^{e_1}$ satisfy $p_{min} \leq x_1^{e_1} \leq v_1 \leq x_1^{e_2}$. And it follows from (67) that the following holds true.

\[
\lim_{m \rightarrow \infty} H(x(t_m)) = \lim_{m \rightarrow \infty} H(x(t_{m+1})) = L_2(p_{min}) = g_2(p_{min} - x_1^{e_2})^2,
\]
(68)

where $p_{min}$ and $g_2$ are same as in (52) and (7), respectively.

By the squeeze theorem one obtains from (67) and (68) that
\[
\lim_{t \rightarrow \infty} H(x(t)) = L_2(p_{min}) = g_2(p_{min} - x_1^{e_2})^2,
\]
(69)

where $p_{min}$ and $g_2$ are same as in (52) and (7), respectively.

As $m \rightarrow \infty$ and all the horizontal ordinates $x_{1,m}$ of switching states $x_m$ are contained in the interval as follows.

\[
(p_{max}, +\infty) \subset (x_1^{e_2}, +\infty), \quad \text{where} \quad p_{min} \leq x_1^{e_1} \leq v_1 \leq x_1^{e_2} \leq p_{max}.
\]

One knows that
\[
\lim_{m \rightarrow \infty} H(x(t_m)) = \lim_{m \rightarrow \infty} H(x(t_{m+1})) = L_1(p_{max}) = g_1(p_{max} - x_1^{e_1})^2.
\]
(70)

It is obvious from (66) and (70) and the squeeze theorem that
\[
\lim_{m \rightarrow \infty} H(x(t)) = L_1(p_{max}) = g_1(p_{max} - x_1^{e_1})^2,
\]
(71)

where $p_{max}$ and $g_1$ are same as in (52) and (7), respectively.

Since $L_1(p_{max})$ and $L_2(p_{min})$ are contained in the region $\Omega$, it can be seen from (19), (67), (68), and (70) that
\[
\lim_{t \rightarrow +\infty} d(x(t), \Omega) = 0.
\]
(72)

(S2) As $v_2 \in (x_1^{e_2}, +\infty)$ and the trajectory $x(t) \in \mathbb{D} - \Omega$, for any $t \in [t_m, t_{m+1})$, $m \in \mathbb{N}$, the switching maximum energy sequence $H(x_m)$ corresponding to switching states $x_m = [x_{1,m}, x_{2,m}]^T \in \mathbb{R}^2$, $m \in \mathbb{N}$, is as follows.
\[
H(x_m) = \begin{cases} 
L_2(x_{1,m}) = g_2(x_{1,m} - x_1^{e_2})^2, \quad \text{as} \quad x_{1,m} \in (-\infty, v_1) \cup (v_2, +\infty), \\
L_1(x_{1,m}) = g_1(x_{1,m} - x_1^{e_1})^2, \quad \text{as} \quad x_{1,m} \in [v_1, v_2], \quad \text{for all} \ m \in \mathbb{N},
\end{cases}
\]
(73)

where the two functions $L_1(\cdot)$ and $L_2(\cdot)$ are defined in (6). $g_1$ and $g_2$ are in (7).

Similar to that proof of Situation (S1), one obtain from Lemma 2 and the condition of $R_1 > 0$ and $R_2 > 0$, (6), (33), (34), (67), and (38) that (72) also holds for this situation. It thus follows from the above considerations of Situations (S1) and (S2) that Statement (ii) holds for Case (a).

As for Case (b), it is easy to see from the conditions of Statement (ii) that system (1) under the switching line $l_1$ in (2) is region stable with respect to the region $\Omega$ defined in (6). Meanwhile, (23) of Definition 3 follows from the fact that the system trajectory $x(t)$ is always contained in the region $\Omega$. Thus, all the conditions of Definition (3) are satisfied for system (1). Therefore, the correctness of Statement (ii) follows from Definition (3) for Case (b).
Finally, we show that Statement (ii) also holds for Case (c) as follows. Similar to that proof of the two cases (C1) and (C2) of Situation (i), one can show from (63) that Statement (ii) holds true for the case that the whole trajectory \( x(t; l_m, x_m, l_m) \) of system (1) is contained in \( \mathbb{D} - \text{int}(\Omega_e) \). On the other hand, it is obvious that as \( \varepsilon \to 0 \), the state trajectory \( x(t) \) contained in the region \( \Omega_1 - \Omega \) will gradually converge into the region \( \Omega \).

Based on the above, by Definition 3 we obtain from the condition of \( R_1 > 0 \) and \( R_2 > 0 \), (27), (68), (70) and (72) that system (1) under the switching line \( l_1 \) in (2) is asymptotically region stable with respect to the region \( \Omega \). That is, Statement (ii) holds.

(2) We show the conclusion of Theorem 1 also holds for the case that \( g_1 = g_2 \).

In this case, letting \( L_1(x_1) = L_2(x_1) \) together with (6) yields

\[
2(x_1^1 - x_1^2)x_1 + (x_1^1)^2 - (x_1^2)^2 = 0. \tag{74}
\]

Solving the above equation (74) we obtain its solution denoted by \( v_1 \) as follows.

\[
v_1 = \frac{x_1^1 + x_1^2}{2}. \tag{75}
\]

It can be obtained from (75) that \( p_{\text{max}} = 2x_1^1 - x_1^2 \) and \( p_{\text{min}} = 2x_1^1 - x_1^2 \) being the two points on the boundary of the region \( \Omega \). Then, similar to that of the case of \( g_1 \neq g_2 \), it can be showed that Theorem 1 also holds for the case that \( g_1 = g_2 \).

Thus, the proof of Theorem 1 is finished.

For the case that horizontal ordinates of the two subsystems’ equilibrium points \( x^e_1 \) and \( x^e_2 \) are same but their vertical ordinates are different, i.e., \( x_1^e = x_2^e \) and \( x_2^1 \neq x_2^2 \), we present another main result of this paper as follows.

**Theorem 2** Consider system (1) with the compact region \( \Psi \) in (20) and the switching line \( l_1 \) in (2). For the case that \( x_2^1 \neq x_2^2 \), system (1) under the switching line \( l_1 \) in (2) is

(i) region stable with respect to the region \( \Psi \) in (20), if \( R_1 \geq 0, R_2 \geq 0, \) and

\[
\frac{\beta}{\alpha} \geq \max \left\{ 1, 4 \left( \frac{x_2^2 - x_1^1}{v_2 - x_2^1} \right)^2, 4 \left( \frac{x_2^2 - x_1^1}{v_2 - x_2^2} \right)^2 \right\}, \tag{76}
\]

where \( \alpha = \min \left\{ \lambda_{\min}(Q_1), \lambda_{\min}(Q_2) \right\} \), \( \beta = \max \left\{ \lambda_{\max}(Q_1), \lambda_{\max}(Q_2) \right\} \), and \( v_2 \) is the intersection point of the two parabolic curves \( M_1(x_2) \) and \( M_2(x_2) \) in (8) over the interval \( (x_2^1, x_2^2) \) in the vertical axis, i.e.,

\[
v_2 = \begin{cases} 
\frac{(\delta_1 x_1^2 - \delta_2 x_2^2) + \sqrt{\delta_1 \delta_2 (x_1^2 - x_2^2)^2}}{\delta_2 - \delta_1}, & \text{as } \delta_1 \neq \delta_2, \\
\frac{x_1^2 + x_2^2}{2}, & \text{as } \delta_1 = \delta_2.
\end{cases} \tag{77}
\]

where \( \delta_1 \) and \( \delta_2 \) are defined in (9).

(ii) asymptotic region stable with respect to the region \( \Omega \), if both the matrices \( R_1 > 0 \) and \( R_2 > 0 \) and the condition (76) are all satisfied.

**Proof:** Similar to that proof of Theorem 1, we show from \( x_2^1 \neq x_2^2 \) that Theorem 2 also holds via replacing all the horizontal ordinates \( x_1^1 \) and \( x_1^2 \) of the two subsystems’ equilibrium points \( x^e_1 \) and \( x^e_2 \) by their vertical ordinates of \( x_2^1 \) and \( x_2^2 \) during the whole proof of Theorem 1.

**4. NUMERICAL SIMULATIONS**

This section is to carry out three numerical examples to verify the effectiveness and practicability of Theorems 1 and 2 obtained in Section 3.
**Example 1** Consider a switched linear Hamiltonian system with two subsystems as follows.

\[
\begin{cases}
\dot{x} = [J_i - R_i] \nabla H_i(x), & \text{for all } t \geq 0, \ i = 1, 2, \\
x(0) = x_0,
\end{cases}
\]

(78) governed by a switching path \( \sigma(t) : [0, +\infty) \to \{1, 2\} \), which is also a switching line passing through the two points, \( x^{e_1} = [1 \ 2]^T \) and \( x^{e_2} = [8 \ 16]^T \), of the equilibrium points of the first subsystem and the second subsystem, respectively. That is, the switching line can be expressed as

\[ l_1 : x_2 = 2x_1. \]

(79) In system (78), the two subsystems’ Hamiltonian functions are listed as

\[ H_i(x) = 0.5(x - x^{e_i})Q_i(x - x^{e_i})^T \text{ for } i = 1, 2, \]

(80) where the matrices \( Q_1 \) and \( Q_2 \) are respectively listed as the subsystems’ corresponding matrices are as follows.

\[ Q_1 = \begin{bmatrix} 18 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} 15 & 0 \\ 0 & 1 \end{bmatrix}. \]

(81) The corresponding matrices of the first and second subsystems are as follows.

\[ J_1 = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & -12 \\ 12 & 0 \end{bmatrix}, \quad \text{and} \quad R_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}. \]

(82)

We obtain from (81) and (82) that

\[ [(J_1 - R_1)Q_1][(J_2 - R_2)Q_2] = \begin{bmatrix} 1800 & 656 \\ -9000 & -1726 \end{bmatrix}. \]

(83) and

\[ [(J_2 - R_2)Q_2][(J_1 - R_1)Q_1] = \begin{bmatrix} 1512 & 504 \\ -9864 & -1438 \end{bmatrix}. \]

(84)

From (83) and (84) it is obvious that although the two subsystems of system (78) are essential two linear systems, their state matrices \( (J_1 - R_1)Q_1 \) and \( (J_2 - R_2)Q_2 \) cannot commute. Therefore, the region stability of system (78) cannot be verified by the stability criteria obtained in the reference\textsuperscript{[13]}. Moreover, there is not any stability criteria reported in the open literature. However, we can check the stability of system (78) as follows.

According to (7), (81), and (36), we obtain that

\[ \alpha = \min \{ \lambda(Q_1), \lambda(Q_2) \} = 1, \quad \beta = \max \{ \lambda(Q_1), \lambda(Q_2) \} = 18, \]

(85) and

\[ v_1 = \max \left\{ 1, 4 \left( \frac{x^{e_2} - x^{e_1}}{v_2 - x^{e_1}} \right)^2, 4 \left( \frac{x^{e_2} - x^{e_1}}{v_2 - x^{e_2}} \right)^2 \right\} = 4.3718. \]

(86)

It is obvious from (85) and (86) that (36) of Theorem 1 is satisfied.

It is obvious from (82), (85), and (86) that \( R_1 > 0, R_2 > 0, \) and (36) are all satisfied for system (78) under the switching line \( l_1 \) denoted in (79). That is, all the conditions of Statement (ii) of Theorem 1 are satisfied for system (78). By Theorem 1 we know that system (78) under the switching line \( l_1 \) in (79) is asymptotically region stable with respect to the following region:

\[ \Omega = \left\{ z \in \mathbb{R}^2 : H_1(z) \leq \max_{i=1,2} \{ H_i(x^{e_i}) \} \right\} \bigcup \left\{ z \in \mathbb{R}^2 : H_2(z) \leq \max_{i=1,2} \{ H_i(x^{e_i}) \} \right\} \]

\[ = \left\{ z \in \mathbb{R}^2 : H_1(z) \leq 539 \right\} \bigcup \left\{ z \in \mathbb{R}^2 : H_2(z) \leq 465.5 \right\}. \]

(87)
To show the above conclusion by simulations, we choose the following two initial states: $x_{01} = [-25 50]^T$ and $x_{02} = [4 8]^T$. It is easy to see that the two initial states $x_{01}$ and $x_{02}$ are respectively contained in the interior and the exterior of the region $\Omega$ in (87). The numerical simulations are then carried out, and the results are shown in Figures 1-4, which denote the trajectories of system (78) under the switching line $l_1$ in (79) starting from the two initial states in the plane $\mathbb{R}^2$, the trajectories with respect to time $t$, and switching path with respect to time $t$, respectively. It can be seen from Figs. 1 and 2 that the trajectories $x(t)$ of system (78) goes to/into the region $\Omega$ as $t \to +\infty$. Therefore, these simulations show that Theorem 1 is effective and practical.

**Figure 1.** The trajectory of system (78) under the switching path $\sigma_1$ relation to the switching line $l_1$ in the plane $\mathbb{R}^2$ starting from the initial state $x_{01}$.

**Figure 2.** The trajectory of system (78) under the switching path $\sigma_2$ relation to the switching line $l_1$ in the plane $\mathbb{R}^2$ starting from the initial state $x_{02}$. 

![Figure 1](image1.png)

![Figure 2](image2.png)
Figure 3. The trajectory of system (78) under the switching path $\sigma_1$ relation to the switching line $l_1$ starting from the initial state $x_{01}$ with respect to time $t$.

Figure 4. The trajectory of system (78) under the switching path $\sigma_2$ relation to the switching line $l_1$ starting from the initial state $x_{02}$ with respect to time $t$. 
Example 2 Consider a switched linear Hamiltonian system with two subsystems as follows.

\[
\begin{align*}
\dot{x} &= [J_i - R_i] \nabla H_i(x), \\
x(0) &= x_0,
\end{align*}
\]

for all \( t \geq 0, \ i = 1, 2, \) (88) governed by a switching path \( \sigma(t) : [0, +\infty) \rightarrow \{1, 2\} \), which is also a switching line passing through the two points, \( x^{e1} = [2 ~ 2]^T \) and \( x^{e2} = [2 ~ 12]^T \), of the equilibrium points of the first subsystem and the second subsystem, respectively. That is, the switching line can be expressed as

\[
l_1 : x_1 = 2.
\]

In system (88), the two subsystems’ Hamiltonian functions are listed as

\[
H_i(x) = 0.5(x - x^{e_i})Q_i(x - x^{e_i})^T \quad i = 1, 2,
\]

(90)
where the matrices $Q_1$ and $Q_2$ are respectively listed as the subsystems’ corresponding matrices as follows.

\[
Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} 15 & 0 \\ 0 & 18 \end{bmatrix}. \tag{91}
\]

The corresponding matrices of the first and second subsystems are as follows.

\[
J_1 = \begin{bmatrix} 0 & -10 \\ 10 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}, \quad \text{and} \quad R_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}. \tag{92}
\]

One obtains from (88), (91) and (92) that the following two:

\[
[(J_1 - R_1)Q_1][(J_2 - R_2)Q_2] = \begin{bmatrix} -17820 & 5976 \\ -9450 & 1260 \end{bmatrix}. \tag{93}
\]

and

\[
[(J_2 - R_2)Q_2][(J_1 - R_1)Q_1] = \begin{bmatrix} -1260 & 17550 \\ -840 & -15300 \end{bmatrix}. \tag{94}
\]

From (93) and (94) it is obvious that although the two subsystems of system (88) are also two linear systems, their state matrices $(J_1 - R_1)Q_1$ and $(J_2 - R_2)Q_2$ are not commutative. Therefore, the region stability of system (88) cannot be verified by the stability criteria obtained in the reference \cite{13}. Moreover, there is not any stability criteria reported in the open literature. However, we can check the stability of system (88) as follows.

According to (7), (91), and (36), we obtain that

\[
\alpha = \min \left\{ \lambda(Q_1), \lambda(Q_2) \right\} = 1, \quad \beta = \max \left\{ \lambda(Q_1), \lambda(Q_2) \right\} = 18, \tag{95}
\]

and

\[
v_2 = \max \left\{ 1, 4 \left( \frac{x_2^2 - x_1^2}{v_2 - x_1^2} \right)^2, 4 \left( \frac{x_2^2 - x_1^2}{v_2 - x_1^2} \right)^2 \right\} = 7.2277. \tag{96}
\]

It is obvious from (92), (95), and (96) that $R_1 > 0, R_2 > 0,$ and 36 are all satisfied for system (88) under the switching line $l_1$ denoted in (89). That is, all the conditions of Statement (ii) of Theorem 2 are satisfied for system (88). By Theorem 2 we know that system (88) under the switching line $l_1$ in (89) is asymptotically region stable with respect to the following region:

\[
\Psi = \left\{ z \in \mathbb{R}^2 : H_1(z) \leq \max_{i=1,2} \{ H_i(x^i) \} \right\} \bigcup \left\{ z \in \mathbb{R}^2 : H_2(z) \leq \max_{i=1,2} \{ H_i(x^i) \} \right\}
= \left\{ z \in \mathbb{R}^2 : H_1(z) \leq 750 \right\} \bigcup \left\{ z \in \mathbb{R}^2 : H_2(z) \leq 980 \right\}. \tag{97}
\]

To show the above conclusion by simulations, we choose the following two initial states: $x_{01} = [-20 \ 50]^T$ and $x_{02} = [2 \ 20]^T$. It is easy to see that the two initial states $x_{01}$ and $x_{02}$ are respectively contained in the interior and the exterior of the region $\Psi$ in (97). The numerical simulations are then carried out, and the results are shown in Figures 7-12, which denote the trajectories of system (88) under the switching line $l_1$ in (89) starting from the two initial states in the plane $\mathbb{R}^2$, the trajectories with respect to time $t$, and switching path with respect to time $t$, respectively. It can be seen from Figs. 7 and 10 that the trajectories $x(t)$ of system (88) goes to/into the region $\Omega$ as $t \to +\infty$. Therefore, these simulations show that Theorem 2 is effective and practical.
Figure 7. The trajectory of system (88) under the switching path $\sigma_3$ relation to the switching line $l_1$ in the plane $\mathbb{R}^2$ starting from the initial state $x_{01}$.

Figure 8. The trajectory of system (88) under the switching path $\sigma_4$ relation to the switching line $l_1$ in the plane $\mathbb{R}^2$ starting from the initial state $x_{02}$. 
Figure 9. The trajectory of system (88) under the switching path \( \sigma_3 \) relation to the switching line \( l_1 \) starting from the initial state \( x_{01} \) with respect to time \( t \).

Figure 10. The trajectory of system (88) under the switching path \( \sigma_4 \) relation to the switching line \( l_1 \) starting from the initial state \( x_{02} \) with respect to time \( t \).
Figure 11. The response of the switching path $\sigma_3$ relation to the switching line $l_1$ with respect to time $t$.

Figure 12. The response of the switching path $\sigma_4$ relation to the switching line $l_1$ with respect to time $t$.

5. CONCLUSIONS

We have studied the region stability of two-dimensional switched linear Hamiltonian systems with multiple equilibrium points. For the case that there are two subsystems and the switching path is a switching line, by the maximum energy function method we have proposed some sufficient conditions of region stability and asymptotic region stability of such kind of switched systems. The stability criteria given are easily-test. Two numerical examples are illustrated the two theorems are effective and practical. Note that the results obtained in this paper are suitable to two-dimensional switched linear Hamiltonian systems with multiple equilibrium points and constraint in a switching line. To remove the above restrictions, the investigation of region stability of high-dimensional switched Hamiltonian systems with multiple equilibria will be our next work.

DECLARATIONS
Authors’ contributions

Made substantial contributions to supervision, writing, review, editing and methodology: Zhu L

Performed writing-original draft, software, validation and visualization: Liu T

Availability of data and materials

Not applicable.

Financial support and sponsorship

This work was supported by Shandong Natural Science Foundation of China under Grant (No. ZR2021MF012) and Cultivating Foundation of Qilu University of Technology under Grant (No. 2022PY1010).

Conflicts of interest

All authors declared that there are no conflicts of interest.

Ethical approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

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